International Journal of Theoretical Physics, Vol. 2, No. 1 (1969), pp. 1–22

On Causal Dynamics Without Metrisation: Part II†

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Abstract

An interpretation of an ordering relation as an operator is constructed by appeal to the notion of homotopy. A metric-free definition of a causal ordering is given, after carefully investigating, by means of the notion of ordered sets, what is crucial to the notion of causality. A metric-free condition is derived that a causal ordering operator must satisfy, and a concrete realisation is found to exist that embodies the paraphernalia of Minkowski space with Zeeman's fine topology. Moreover, it is seen that a coordinate space used to give a cartography to causally related events must have 'at least' a hyperbolic metric.

1. *Introduction*

1.1. *Résumé*

The impossibility of asserting the complete metrisability of the whole of three space, deduced in Part I of this series,‡ gives rise to the question of how one may define causality in the large. The real problem is of how to do so without using any form of metric notion.

A means of solving this problem is presented that uses the notion of ordering operators, a notion derived from that of an ordering relation. But this itself is not sufficient, for a more fundamental study is needed to investigate the basis of the notion of causality. This is achieved by introducing the 'Principle of Corporate Agreement', which proposes an explicit means of objectivising the content of observations. The main use of this principle will be in Part III of this series (this volume, pp. 23–40), but here its application allows one to define a causal ordering as a total ordering of events which are experienced in common by a set of observers.

From a condition (derived in Section 2) that total ordering operators must satisfy, Section 4 shows that there exists a concrete realisation of a total ordering operator that demands use of Zeeman's fine topology for Minkowski space. It is seen, furthermore, that the disconnection, inherent in the fine topology's neighbourhoods, is essential to any realisation of an ordering operator satisfying the metric-free causality condition.

† This work was undertaken and completed whilst the author was at the Post Office Research Station, Dollis Hill, London, N.W.2.

‡ On Causal Dynamics Without Metrisation: Part I. *International Journal of Theoretical Physics*, Vol. 1, No. 1, 115–151 (1968).

1.2. *Prospect*

The interpretation of an ordering relation as an ordering operator is rather intuitive, and can bear a good deal of rigorisation. The next paper in this series is devoted to that task, and by virtue of the more careful expression of a non-metric notion of causality in terms of ordering relations and ordering operators, there is deduced that systems of causal dynamical transitions between events must have a 'quotient structure'.

2. *Ordering Relations*

2.1. *Ordering Axioms*

Before discussion of the ideas behind causality, it will be necessary to recall elementary notions about ordering relations. The notion of ordering is familiar to everyone through the two verbs 'to succeed' and 'to precede'. When one has a large collection of objects in a geometrical pattern it is rather more difficult to give a precise meaning to these terms by means of mere words. Rather one refers to a set of axioms which provide a firm starting point. The reflexive and transitive axioms define a *pre-ordering*, the addition of the anti-symmetric axiom defines a *partial* (or *strict*) *ordering*, and the transitive axiom together with the total axiom define a *total ordering*. An equivalence relation is a variant of a strict ordering, in which the anti-symmetric axiom is replaced by the symmetric axiom. (For clarification on this point, the reader should refer to Section 4.3.) The following definition and theorem, which are very well known, will be of great use to us in the discussions of this and the following paper of the series:

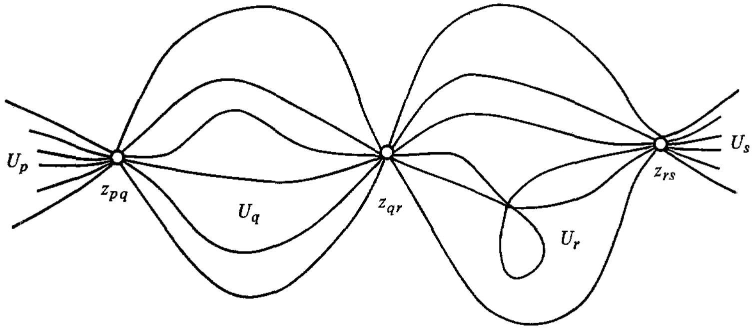
DF(1): Let $\mathscr{C} = \{C_i\}$ be a partition of a set A, then if $a_{\lambda_i}, a_{\mu_i} \in C_i$ for any i, the relation written $A/\mathscr{C}$ such that $a_{\lambda_i} A/\mathscr{C} a_{\mu_i}$ is called the *equivalence relation* induced *by the partition* $\mathscr{C}$.

TH(1): The transition $\mathscr{C} \to A/\mathscr{C}$ is the exact reverse of the transition $R \to A/R$.] Halmos (1958), Section 7, p. 28

This reciprocity relation expresses the connection between equivalence relations and partitions.

2.2. *Bounds of Ordered Sets*

It is possible to separate sets into subsets together with certain elements so that the single elements play a distinctive role in giving an ordering to the sequences of subsets of the complete sets. For example, consider Fig. 1, in which are shown four sets U_p, U_q, U_r, U_s each containing totally ordered paths (sets of elements), with three points common to all of the paths. Let each of the four sets $U_p, \dots, U_s$ be ordered by a common pre-ordering $<$, so that for all $u_p \in U_p, \dots, u_s \in U_s$, there holds $u_p < u_q < u_r < u_s$. Then it is possible to define an ordering relation $\circ\!\!<$ for the set Z in such a manner that it is a total order, even if the intermediate sets are no more than pre-ordered. The ordering relation so defined then stays total under the restric-



$U_p < U_q < U_r < U_s$ $\qquad$ $z_{pq} \circ\!\!< z_{qr} \circ\!\!< z_{rs}$

i.e. $u_p < u_q < u_r < u_s$ for all $u_p \in U_p, u_q \in U_q, u_r \in U_r, u_s \in U_s$.

Figure 1.

tion of the ordering of the $u_n \in \{U_n\}$ to partial or total. It is clear that each element of Z can be described simultaneously as the immediate predecessor of one of the $U_p, \ldots, U_s$ and as the immediate successor of another; in fact the terms sup and inf may be applied. The ordering relation $\circ\!\!<$ is defined as follows:

DF(*2*): Given a collection of sets $\mathscr{U} = \{U_i\}$ together with a pre-ordering $<$ between the elements of $\bigcup_i U_i$ such that not only $U_i < U_j$ for $i < j$† but also $u_{\lambda(i)} < u_{\mu(i)}$ for all $u_{\lambda(i)}, u_{\mu(i)} \in U_i$, and such that $\sup U_p = z_{pq} = \inf U_q$, then the set Z of all elements z_{pq} is ordered by the relation $\circ\!\!<$ in such a manner that if $z_{pq} \circ\!\!< z_{qr} \circ\!\!< z_{rs}$, then:

$$\sup U_p = z_{pq} = \inf U_q$$
$$\sup U_q = z_{qr} = \inf U_r$$
$$\sup U_r = z_{rs} = \inf U_s$$

TH(*2*): The ordered set $(Z, \circ\!\!<)$ is a totally ordered set.

The proof of this theorem, although somewhat lengthy if written in detail, is almost apparent from the definitions of sup and inf.]

2.3. *Pre-order Commutation Relations*

In this and subsequent sub-sections, ordering relations are used as operators in a way which mathematicians would not consider to be absolutely respectable. However, for the sake of the physical notions which will be introduced later, we shall disregard such objections, believing that they may be overcome.

If W is a set with a pre-ordering relation R and $x, y \in W$ are ordered by R—that is to say we may write xRy—then since the strict axiom does not apply it is possible to write both xRy and yRx without requiring x and y to be identical.

† See Fig. 1; i.e. $u_p < u_q < u_r < u_s$ for all $u_p \in U_p$, $u_q \in U_q$, $u_r \in U_r$, $u_s \in U_s$.

Now since

$$xRy, \quad \text{then by inversion of the ordering relation,} \quad yR^{-1}x \quad (2.3.1)$$

also it is true that,

$$yRx, \quad \text{so by inversion of the ordering relation,} \quad xR^{-1}y \quad (2.3.2)$$

Using R and R^{-1} as right operators—in the sense that if $aR^{\pm 1}b$, then $b = (a)R^{\pm 1}$—equation (2.3.1) implies:

$$xRR^{-1}x \quad (2.3.3)$$

whilst equation (2.3.5) implies:

$$xR^{-1}Rx \quad (2.3.4)$$

Therefore, in some sense we may say that $RR^{-1} = R^{-1}R$, or in the more compact notation of commutator brackets we may state:

PR((1)): If R is a pre-ordering (in a set), then $[R, R^{-1}] = 0$.]

RMK(1): The sense of equality is justifiable on these grounds: that the pre-ordering axioms do not differentiate between the two starting relations xRy and yRx—therefore equations (2.3.3) and (2.3.4) would both have been obtained regardless of the (dummy) symbol x. And since no form of differentiation could be made in this respect—which is actually the respect of assigning a direction to R—one may write an equality relation.

Let us note that R and its 'powers' do not combine 'multiplicatively' according to the usual law of addition of exponents. For instance, combining the property of reflexivity, xRx, with equations (2.3.3) and (2.3.4) might lead us to conclude from $R = RR^{-1} = R^{-1}R$ that $R = R^{-1} = I$, that is to say R is no more than the trivial identity relation—a result clearly not necessarily true. Moreover, from the transitivity condition applied to xRy and yRz we obtain xRz; inverting we have both $zR^{-1}R^{-1}x$ and $zR^{-1}x$. Generalising this, we obtain the unusual properties:

$$R^m = R; \qquad R = R^{-1} \not\approx I; \qquad R^{-n} = R^{-1} \quad (2.3.5)$$

Here the sense of the equality relation may be interpreted, 'Does a task similar to', for we have considered R and R^{-1} as operators taking one element of a set into another.

RMK(2): Notice that the inequality in equation (2.3.5) expressed by $R = R^{-1} \not\approx I$ is 'almost' an equality (especially in the sense described), in the sense that a pre-ordering relation provides no means of testing for the identity of elements related by it. In contrast to this, the symmetric axiom of a strict ordering does provide such a means.

Equation (2.3.5) allows us to generalise PR((1)) to the following statement:

PR(1): If a set is pre-ordered by a relation R, then considering R as an operator associating pairs of elements, it satisfies the condition $[R^m, R^{-n}] = 0$.]

Some reason becomes apparent in the composition behaviour of R if it is considered as a non-trivial group of elements. We shall use this interpretation later on.

2.4. *Partial Order Commutation Relations*

The commutation relation of PR(1) may be shown to hold for strict orders also, but the proof needs modification because equations (2.3.1) and (2.3.2) cannot both be written, in view of the symmetric axiom. Instead set xRy, with the inverse relation $yR^{-1}x$, and then choose an element w such that wRx, with inverse relationship $xR^{-1}w$. So we may combine these relationships together in the same way as in the previous section and state:

PR(2): If a set is (strictly) partially ordered by a relation R, then considering R as an operator associating pairs of elements, it satisfies the condition $[R^m, R^{-n}] = 0$. ▌

The composition laws for a partial ordering are essentially those of equation (2.3.5) except that the inequality must be slightly modified. For when two elements of the ordered set satisfy the strict axiom (that xRy and yRx together imply $x = y$), we have:

$$xRy \Leftrightarrow yR^{-1}x, \qquad yRx \Leftrightarrow xR^{-1}y$$

whence:

$$R = R^{-1} = I$$

here the symbol $\Leftrightarrow$ means that each side uniquely implies—and is identical to—the other. Therefore in the case of *general* pairs of elements partially ordered, we may write

$$R^m = R; \qquad R^{-n} = R^{-1}; \qquad I \underset{\circ}{\neq} R \underset{\circ}{\neq} R^{-1} \underset{\circ}{\neq} I \qquad (2.4.1)$$

NTN(1): The symbol $\underset{\circ}{\neq}$ means that the inequality holds in general, except where the ordering relation (operator) carries an element into itself or into another which may be described as 'equal'.

2.5. *Total Order Commutation Relations*

The total axiom demands that certainly xRy be true, or else that yRx be true, but *not* both. First suppose that xRy be true. Then $yR^{-1}x$, and consequently $xRR^{-1}x$. Secondly, if we were to suppose that yRx were also true, then we would have $xR^{-1}Rx$ by employing the supposedly true inverse relation $xR^{-1}y$ and using $y = (x)R^{-1}$. Consequently we could write $RR^{-1} = R^{-1}R$. But our second assumption is false, therefore we must have $[R, R^{-1}] \neq 0$. By using the transitivity axiom we may generalise the result as before, to refer to positive and negative exponents of R:

PR(3): If a set is totally ordered by a relation R, then considering R as an operator associating pairs of elements, it satisfies the condition $[R^m, R^{-n}] \neq 0$. ▌

3. *The Principle of Corporate Agreement*

3.1. *First Considerations*

The study called 'Physics' is a corporate activity, demanding the use and acceptance of some common form of observational activity together with an inferential–interpretative language by a collection of people who may be called observers. Let us begin with considering a single observer and look for some characteristics of his experience of the world about him.

One may say, with a commonsense meaning, that a person experiences a set of events during any given period of time. That is to say a person may acknowledge that in the given interval he did certain things, saw, heard or otherwise sensed certain things (including what may have been done to him by other people), and thought about or thought of certain things. All these apprisable experiences we shall briefly call '*events*'. It is quite clear that in a commonsense way one may speak of an observation of a physical system, carried out by a person, as an event in the person's experience of the world about him—for his action of observation requires determined† actions, perceptions and thoughts, and these are qualities customarily ascribed to the description of the experiencing of an event.

Suppose that an observer experiences a set of events E. Then if he has any more detail in his memory other than the nature of the different events, he will be able to ascribe some sequential order to the elements of E. Of certain pairs of elements he will be able to say that this one immediately succeeded that one; it is not certain that his memory, or perhaps his sense of distinguishing between events, will be sufficiently appropriately developed to allow him to assign an immediate predecessor or successor to every single event. More generally, it is clear that he will be able to establish some semblence of a 'before–after' relationship between elements or between subsets of E if and only if it is possible to find at least one element of E which, with respect to the 'before–after' ordering relation, possesses either a strict initial segment or strict 'subsequent' segment.‡ In day-to-day life one does in fact find that there are collections of events which cannot be precisely ordered amongst themselves in the memory, but nevertheless it is possible to say that the particular subset of events succeeded one particular event and preceded another:¶ hence the caution of the previous sentence.

† 'Determined' is not used here in any mathematical sense, but in the sense that the 'ego-self' of the observer must decide to involve itself with the actions necessary, on its part, for the observation to be undertaken. (For an excellent discussion of the relationship between man's apprehension of his circumstances and his awareness of his action of apprehension, the reader is referred to Karl Heim's work, *Christian Faith and Natural Science*, SCM Press, London, 1953. In particular, the chapter entitled 'The Ego and The World' is relevant to the study here, especially the first and second existential propositions.)

‡ Compare with the notion of the strict initial segment with respect to an element in an ordered set. We may call, in an obvious way, the set $\{x \in X : x > a\}$ the strict subsequent segment determined by a.

¶ And later on, just so do I propose that there may be sets of events in the real physical world we perceive, which may not be able to be totally ordered by our present notions of space and time.

Let us now introduce a second observer who experiences a set of events E'; furthermore, let us suppose that they can communicate with one another in a mutually intelligible way. It is then possible that under favourable circumstances the observers will agree that each has observed a number of events which the other has observed. (Notice that in the everyday interpretation of this last sentence there would have been excluded, tacitly, almost any identification of thoughts as events—so accustomed are we to being aware of what occurs *outside* ourselves.) Immediately there arises the question of whether each observer puts the common events into the same (perhaps totally) ordered sequence as the other.†

If two observers cannot agree upon an ordering of the events common to their observation, there arises the situation in which one observer may say event e_λ preceded event e_μ, whilst the other says that e_λ succeeded e_μ. But there are, of course, certain conditions which must be satisfied before such statements can be meaningfully made, and these will be examined in the following subsections. We shall arrive at the most fundamental question which can be asked about the problem of causality—and it will be shown why it is so fundamental by the discussions in the ensuing text—namely this:

QU(*1*): Given a set of observers, each one of which records a set of events and ascribes at least a strict ordering to it, then if there is a totally ordered non-vanishing intersection of all such sets of events, is it possible to extend that total ordering to a total ordering of the union of all sets of observed events?

3.2. *Direction Comparisons*

Consider two observers $\mathcal{O}_1$, $\mathcal{O}_2$, who record sets of events $E^{(1)}$, $E^{(2)}$, respectively. Let us denote subsets of $E^{(1)}$, $E^{(2)}$ by $E_\lambda^{(1)}$, $E_\mu^{(2)}$, and elements of these subsets by $e_{\lambda(i)}^{(1)}$, $e_{\mu(j)}^{(2)}$, respectively; elements of $E^{(1)}$, $E^{(2)}$ will have the Greek subscript omitted if they refer to no particular subset. Suppose also that $E^{(1)} \cap E^{(2)} = E_\Gamma \neq \varnothing$, so that $\mathcal{O}_1$ and $\mathcal{O}_2$ both observe the set of events E_Γ. If either observer is to be able to state that a particular event which he observes is the cause (or a possible cause) of another particular event which he observes, then it must be possible for him unambiguously to distinguish each of the two events by means of an ordering relation, in the sense: in the domain of the relation it is possible to separate the set of events into two disjoint subsets, one containing the first mentioned event and the other containing the second event. A pre-ordering does not allow any such unambiguous distinction; a partial ordering generates equivalence classes,‡ members of any one class being indistinguishable from one another (i.e. equivalent) as far as the ordering relation is concerned; a total ordering completely separates the ordered elements into a sequence isomorphic to a naturally ordered subset of the integers. Therefore, in order for an observer

† For example, husband–wife disputations over who did what, first!!

‡ A partial ordering is not an equivalence relation, but see Section 4.3 for a proper discussion of the way in which equivalence classes may be generated and compounded by a partial ordering.

to be able to associate a well-defined cause–effect relationship with a set of events, he must be able to introduce a total order into the set. If each observer's cause–effect relationship between certain elements of E_Γ is to have a common meaning, further conditions must be satisfied.

Let $e^{(i)}_{\Gamma[I]}$ be $\inf E_\Gamma$ for $E^{(i)}$, $i = 1, 2$: (this notation is introduced in case each observer considers a different element of E_Γ to be the infimum with respect to his ordering). To each $E^{(i)}$ there will be a subset $E^{(i)}_{\Gamma[T]} \subseteq E_\Gamma$ which is totally ordered by a relation $R^{(i)}_{\Gamma[T]}$. Let $e^{(i)}_{\Gamma[S]}$ be $\sup E_\Gamma$ for $E^{(i)}$. Then there will stand the relations $e^{(i)}_{\Gamma[I]} R^{(i)}_{\Gamma[T]} e^{(i)}_{\Gamma[S]}$, $i = 1, 2$, but these two relations only establish a relation between the directions of $R^{(1)}_{\Gamma[T]}$ and $R^{(2)}_{\Gamma[T]}$ if the two pairs of elements $e^{(1)}_{\Gamma[I]}$, $e^{(1)}_{\Gamma[S]}$ and $e^{(2)}_{\Gamma[I]}$, $e^{(2)}_{\Gamma[S]}$ consist of the same two elements. In the completely general case, there is no reason to suppose that this may be so. However, if $e^{(1)}_{\Gamma[I]} = e^{(2)}_{\Gamma[I]}$ (hence $e^{(1)}_{\Gamma[S]} = e^{(2)}_{\Gamma[S]}$, because of the total ordering) we can say that $R^{(1)}_{\Gamma[T]}$ defines the same direction as $R^{(2)}_{\Gamma[T]}$; in the other case when $e^{(1)}_{\Gamma[I]} = e^{(2)}_{\Gamma[S]}$ (hence $e^{(1)}_{\Gamma[S]} = e^{(2)}_{\Gamma[I]}$) we say that the directions of the $R^{(i)}_{\Gamma[T]}$ are opposite. The general case is now seen to require a more detailed treatment; we must search for the first pair of elements of $E_{\Gamma[T]}$ that are ordered in the same direction and such that both elements are predecessors of $e^{(1)}_{\Gamma[S]}$ and $e^{(2)}_{\Gamma[S]}$ in $E^{(1)}_{\Gamma[T]}$ and $E^{(2)}_{\Gamma[T]}$ respectively. This means that there must exist at least three elements of $E_{\Gamma[T]} = E^{(1)}_{\Gamma[T]} \cap E^{(2)}_{\Gamma[T]}$, say $e^{(j)}_{\Gamma[T]}$, $j = 1, 2, 3$, satisfying $e^{(i)}_{\Gamma[I]} R^{(i)}_{\Gamma[T]} e^{(j)}_{\Gamma[T]}$, $e^{(j)}_{\Gamma[T]} R^{(i)}_{\Gamma[T]} e^{(j+1)}_{\Gamma[T]}$ and $e^{(j)}_{\Gamma[T]} R^{(i)}_{\Gamma[T]} e^{(i)}_{\Gamma[S]}$, $i = 1, 2$. These elements are needed so that the transitivity axiom may be used on each $R^{(i)}_{\Gamma[T]}$ to establish a direction 'along' the whole sets $E^{(i)}_{\Gamma[T]}$. Only then can the direction of an arbitrary ordered pair $e^{(1)(i)}_{\Gamma[T]} R^{(1)}_{\Gamma[T]} e^{(1)(j+m)}_{\Gamma[T]}$ be compared with the direction of another arbitrary pair $e^{(2)(k)}_{\Gamma[T]} R^{(2)}_{\Gamma[T]} e^{(1)(k+n)}_{\Gamma[T]}$, and the result tested for consistency with the relation between the directions given by the three test events. There is clearly the possibility that certain pairs of events will give inconsistent comparisons, in which case $\mathcal{O}_1$ and $\mathcal{O}_2$ will be unable to assign a common total order to $E^{(1)}_{\Gamma[T]} \cap E^{(2)}_{\Gamma[T]} = E_{\Gamma[T]}$. Therefore if there is to be any cause–effect relationship which has the same meaning to $\mathcal{O}_1$ and $\mathcal{O}_2$ together, it can only be amongst the set $E^{(T\max)}_{\Gamma[T]} \subset E_{\Gamma[T]}$ consisting of the maximum number of elements which are totally ordered into the same sequence by both $R^{(1)}_{\Gamma[T]}$ and $R^{(2)}_{\Gamma[T]}$.

3.3. *Causality for Several Observers*

The extension of the considerations of the previous subsection make it clear that if a set of observers $\mathfrak{O} = \{\mathcal{O}_i\}$, $i \in I \subset J^{(+)}_\infty$ each individually observe sets of events $E^{(i)}$, then a common cause–effect relationship will be defined if in $E_\Gamma = \bigcap_i E^{(i)}$ there lie sets of events $E^{(i)}_{\Gamma[T]}$ totally ordered for each observer, and it is possible to find a maximal subset $E^{(T\max)}_{\Gamma[T]} \subset E^{(i)}_{\Gamma[T]}$ for all $i \in I$ that is given the same total ordering $R^{(\max)}_{\Gamma[T]}$ (up to a reversal) by every $R^{(i)}_{\Gamma[T]}$. It is by this means that a definition of the notion of a causal ordering of events will be given in Section 3.6, DF(3). But we shall enter into some more careful considerations of the validity of this approach first of all.

3.4. *Zorn's Lemma*

It is necessary to examine the existence of solutions to the problem (of determining a total order of a certain set) that has been posed. For if, as in the previous subsection, there are a large number of sets $E^{(i)}$, the union $E \equiv \bigcup_i E^{(i)}$ will contain a large number of subsets which are pre-ordered, partially ordered and totally ordered by many independent ordering relations. It is not at all clear, at first sight, that it may be possible to define a means of establishing *any* kind of total ordering relation for the whole set.

First consider individual sets $E^{(i)}$. Each observer $\mathcal{O}_i$ will be able to separate the events into subsets which will consist of collections of pre-ordered events (denoted by $E^{(i)}_{P(\lambda)}$ with $\lambda \in \Lambda \subset I$) and strictly ordered events (denoted by $E^{(i)}_{S(\mu)}$ with $\mu \in M \subset I$ and $\Lambda \cap M = \phi$). Furthermore it is possible to choose the $E^{(i)}_{P(\lambda)}$ so that each whole set may be totally ordered† with respect to another and with respect to the $E^{(i)}_{S(\mu)}$. This type of ordering may be compared pictorially with a string of sausages in which the necks between the sausages may have various numbers of knots; the sets of points in the sausages represent the $E^{(i)}_{P(\lambda)}$ and the sets of knots in the necks represent the $E^{(i)}_{S(\mu)}$. It is perhaps more elegant to call a totally ordered set a chain rather than continually refer to sausages and knotted necks; so we may remark that the $E^{(i)}$ may be formed into chains; let us denote the chain forms of $E^{(i)}$ by $\mathscr{C}[E^{(i)}]$. Now divide each $\mathscr{C}[E^{(i)}]$ into subsets which are also chains, in which each element is either an immediate predecessor or an immediate successor of another element, or both; furthermore require that the initial and final elements be drawn from some $E^{(i)}_{S(\lambda)}$—λ must be different for each of these two points if these subsets contain an $E^{(i)}_{P(\lambda)}$ as an element—and denote the subsets as $\mathscr{C}_s[E^{(i)}]$. It is not necessary to require that $\mathscr{C}_s[E^{(i)}] \cap \mathscr{C}_{s'}[E^{(i)}] = \varnothing$. Then every $\mathscr{C}_s[E^{(i)}]$ has well-defined upper and lower bounds. Consequently the conditions of Zorn's Lemma are satisfied‡ —where in fact only an upper bound for every chain is required—and we are assured of the existence of a maximal element of $\bigcup_{i,s} \mathscr{C}_s[E^{(i)}] \equiv \mathscr{C}'[E] \subset \mathscr{C}[E] \equiv \bigcup_i \mathscr{C}[E^{(i)}]$. Notice that from the way the $\mathscr{C}_s[E^{(i)}]$ have been defined, the difference $\mathscr{C}[E] - \mathscr{C}'[E]$ will always be an integral number of sausages (possibly zero); that is to say $\mathscr{C}[E] - \mathscr{C}'[E] = \bigcup_{\lambda \in \Lambda'} E^{(i)}_{P(\lambda)}$, $\Lambda' \subset \Lambda$. The maximal element may belong to $\mathscr{C}[E] - \mathscr{C}'[E]$.

Let us now apply this argument iteratively. Denote by $\mathscr{C}^{(1)}[E]$ the smallest subset of $\mathscr{C}[E]$ such that there exist a maximal element e_1 of the set $\mathscr{C}^{(1)\prime}[E]$ in $\bigcup_i \bigcup_{\lambda,\mu} (E^{(i)}_{P(\lambda)} \cup E^{(i)}_{S(\mu)})$. Then let $\mathscr{C}^{(2)}[E]$ be the next largest subset of $\mathscr{C}[E]$ such that there exist a maximal element e_2 of the set $\mathscr{C}^{(2)\prime}[E] \supset \mathscr{C}^{(1)\prime}[E]$, with $e_2 \neq e_1$. Then e_2 is necessarily an immediate successor of e_1 with respect to this ordering (defined by application of Zorn's Lemma). Let $\{e_\gamma\}$ be the set of all such maximal points. Then if we

† Since we are here dealing with discrete sets of events, we may consider a strict order to define a total order in an unambiguous way; the equality may be dropped from the strict ordering because of the discreteness of the events.

‡ In fact slightly more so, because we may assume $\mathscr{C}[E^{(i)}]$ to be totally ordered, whilst Zorn's Lemma requires no more than strict ordering.

take $\mathscr{T}[E] = \{e_\gamma\} - (\{e_\gamma\} \cap (\bigcup_{i,\lambda} E^i_{P(\lambda)}))$, the set $\mathscr{T}[E]$ is a set of *distinguished* events totally ordered by every observer into the same sequence.

3.5. *Well Ordering*

The Well Ordering Theorem (Halmos, 1958) states that every set can be well ordered. That is to say in this case E can be partially ordered† and every non-empty subset of E can be assigned a smallest element. However, there is in fact no guarantee that the resulting well order will have any similarity—in the appropriate domains—to any of the orderings associated with the observers $\mathcal{O}_i$. This theorem is therefore of little use to our problem, for we need to express any general total ordering of events in a way that is assured of the corporate agreement of the set of observers.

3.6. *Principle of Corporate Agreement*

Although in the last two subsections it has been shown that there is a formal mathematical basis for assuming that under certain circumstances a collection of observers may be able to establish a common causal ordering amongst some events, no useful construction has been given. In this subsection a certain structure is proposed and it is shown to give rise to an intuitively comprehensible diagrammatic technique that may be used for discussing problems of causal ordering; all of this is dependent upon a principle which I shall call the Principle of Corporate Agreement. A brief explanation of its basis is now given.

In attempting to be an objective study, physics endeavours to discuss its subject matter in a way which is as independent as possible from the subjective interpretations of individual participants in the study (whom we shall call 'physicists'). It is the purpose of this present paper to indicate a mathematical formalism which expresses this intent; it is not the case that the expression is tacit within the formalism, but rather that the formalism provides a direct statement of the intent. To achieve the maximum reduction in subjectivity, each 'physicist's' experiences must be reproducible within all other 'physicists': and this tacitly requires that:

ASSN(10): (Principle of Corporate Agreement): *There exists a theory language‡ which*

CDN(1): All 'physicists' use to describe and inter-relate their experiences;

CDN(2): Each 'physicist' believes all the other 'physicists' interpret and use in exactly the same way as he does.

Notice that this principle cannot be entirely objective because it depends upon the subjective act of agreement. However, it does reduce the introduc-

† Since the human action of observation is one that generates discrete observations, the set E may be considered discrete, and hence be considered as totally ordered.

‡ Here one may omit the adjective 'theory'. The notion of 'theory language' is introduced in the discussions of Part III of this series. For the sake of simplicity we shall merely remark that it refers to a collection of symbols that have the same (mathematically) well-defined usage and grammar.

tion of subjectivity to the level that is both inexpressible by the language and not recognised, nor accounted for, by all 'physicists', in the same way.

For the diagrammatic construction we refer back to Fig. 1, and for the associated ordering relations to DF(2), TH(2), at the end of Section 2.2. It is clear that the individual lines in Fig. 1 may be identified as the sets of observations (ordered into at least a partial sequence) corresponding to individual observers. The elements of the set Z are exactly the events upon which all observers can both agree in description† and agree as to total ordering. The latter agreement is assured by the sup and inf conditions in DF(2) as well as TH(2). We may presume that the observers (i.e. 'physicists') are able to set up a total ordering relation for the set Z by means of their common language. The problem then arises of whether that total ordering can be extended to include all the intermediate events between the elements of Z: i.e. if a well ordering for one intermediate set can be found, can it be extended to all the others in a consistent way? This particular problem will be completely avoided (see Section 3, Part III), by taking as of physical interest only those transitions between events upon which all observers can agree. That requirement will be expressed by, as it were, dividing out all the intermediate transitions upon which there is some disagreement. Furthermore, the quotient operation will be taken as a requirement of the Principle of Corporate Agreement.

The diagrams may be given a stricter interpretation than the rather conversational one which has so far been used by replacing events with neighbourhoods of parameter spaces of measurement values, and by replacing the connecting lines between such events by dynamical mappings. Then via inverses of measurement processes, the diagrams correspond to sets of physical conditions and dynamical processes. Consequently the separation of sets of intermediate events into classes, defined by an intermediate event being common to some (but not all) sets of observations, suggests that one might expect to find dynamical processes between certain pairs of physical conditions also separated into classes. The 'algebraic'-set-theoretic substantiation of this is the object of Part III, in which this paper's notions will be taken up once more. However, we may finally give a definition of a causal ordering in accordance with the Principle of Corporate Agreement:

DF(3): A set of events is said to be causally ordered with respect to a set of observers, if each observer can assign a total order to the set that orders the set into a totally ordered sequence coinciding with a total ordering assigned by each of the other observers. All such total orderings are said to induce a common *causal ordering* of the set of events.

† In case the reader is not too sure whether 'agreement of description' has any real meaning, he may be referred to Section 2.3, DF(1), in Part III of this series, where the notion of 'language equivalence' is introduced in order to consider such criteria.

4. *The Representation of a Causal Ordering*

4.1. *Homotopy Considerations*

Let us start by considering pre-orderings. An example of a pre-ordered set is a metrisable space; for if a point x stands in the metric relationship with another point y,† writing this xRy, then if also there holds yRz for the points y, z, then xRz is true also, thus satisfying the transitivity condition. However, one may also write zRy and yRx, hence zRx, without having the points x, y, z identical. This general relationship covers every conceivable metric separation between two points, therefore in the connected components of the metrisable space one may think of the metric relation as being 'stretched' in accordance with the metric separation. Such a notion of stretching is very much like that of homotopic deformation of a space or mapping.

A more complicated relation, on an appropriately topologised set, that is still a pre-ordering, is the (semi-definite) Euclidean metric. It is much more apparent that this relation has homotopy-like properties, in so far as its form for mapping two close-lying points into a number may be smoothly deformed to map two widely separated points onto another number. Now although these properties are not strictly homotopy properties, we may see that there is a very close correspondence with the approach developed in the last two sections. The ordering relation operators take one element of a set into another, so if we consider the simple example of a straight line ordered by the Euclidean metric, then any point on the line can be mapped into another in the following way:

EX(1): Let X be the real line and denote the Euclidean metric by

$$\mathscr{S}(x_i, x_j) = +\sqrt{[(x_i - x_j)^2]} = s_{ij} \qquad \text{for } x_i, x_j \in X$$

Notice that the squaring operation leads to a possible identification of two points; but this may be avoided by the use of a step function. Then if x_i is a fixed point and s_{ij} is a real number, there exists a point x_j at distance s_{ij} from x_i. We may write this in an operator notation as $x_j = \mathscr{S}[x_i; s_{ij}]$, and more explicitly as:

$$x_{j(\pm)} = \mathscr{S}_{(\pm)}[x_i; s_{ij}] = \int dx(x - x_i) \times \delta(l(x; x_i))$$

where $l(x; x_i) = (x - x_i)^2 - s_{ij}^2 \Theta(\pm(x - x_i))$ and where the subscript $(\pm)$ refers to the cases $(x_i - x_j) \gtrless 0$, respectively.

The integral expression is quite clearly a one parameter homotopy of mappings, with parameter s_{ij}.

One may laboriously extend the example above to give analytic expressions for homotopies of mappings for other spaces more complicated than the real line. The general process may, however, be described quite simply.

† Here we do not specify that any numbers are involved, merely that xRy means y is metrically related to x. Hence the symmetric axiom does not hold, because xRy and yRx are both true without implying $x = y$.

From each point of a space draw a path to another point, in such a way that no point is inaccessible from at least one other point of the space. Then the collection of such paths map the space onto itself, and there are as many different mappings of such a nature as there are possible assignations of paths between all pairs of points. If, as is true in a metric space, the lengths of the paths are continuously variable,† then the collection of endomorphisms (of the space) we have described can be arranged into a homotopy of endomorphisms. One can directly consider this homotopy of endomorphisms to be an ordering operator, which (by choice of the correct parameters) can map any one point of the space into any other by any path consistent with the structure‡ of the space. Henceforward, it will be assumed that an ordering operator may relate any two points of a space in a manner consistent with both the structure of the space and the stringency of the ordering (pre-, strict or total).

4.2. *Homotopy of Pre-ordering Commutators*

The equations we have to interpret are PR(1) and equation (2.3.5) of Section 2.3. Taking the simplified form of the commutator, as expressed by $[R,R^{-1}] = RR^{-1} - R^{-1}R = 0$, we may interpret the term RR^{-1} as a double operation upon a set of elements that maps an element a on the left into another (intermediate) one c related to it by the ordering relation R and then transforms it into an element b which also stands in the relation R to the aforementioned intermediate element c. Thus one obtains all possible maps (denoted by RR^{-1}) between ordered pairs of elements (a,b) such that both elements stand in the relation R to an element c. These maps may be identified with paths ($a \to b$, $b \to c$) with an arrowhead.¶ The term $R^{-1}R$ likewise gives rise to a set of arrowhead paths, but with an opposite sense corresponding to the difference in the order of the R and R^{-1} operators. The commutator relation states that the two sets of maps are indistinguishable; in the sense of the preceding subsection, the two sets are (homotopically§) deformable into one another and possess no (homotopy§) classes which cannot be deformed one into the other (because of the weakness of the pre-ordering axioms). We may therefore say that:

PR(4): The set of ordering operators associated with a pre-ordering of a set is simply connected.

This is doubly evident when one inspects the proof of PR(1), for there the arrowhead maps an element into itself, that is to say they correspond to loops. And if loops on a space are all deformable one into another the space is then called simply connected.] The commutator relation also

† To say the space is locally arcwise connected would be more succinct.

‡ Structure for spaces has not yet been defined, but the reader is left to his intuition for the moment. If he thinks about paths and holes he will have the right kind of ideas.

¶ That is to say the directed paths $a \to c$ and $b \to c$ may be drawn as an arrowhead with c as the tip.

§ Using this term firstly in a figurative sense, and secondly because in mathematics we consider smooth deformations to have the form of a homotopy transformation.

implies that an unambiguous direction cannot be assigned to the 'loop' maps.

4.3. *Equivalence Classes and Partial Orderings*

An equivalence relation R in a set satisfies the pre-ordering axioms, but has the symmetric axiom, namely $xRy \Leftrightarrow yRx$, whereas a partial ordering P has the anti-symmetric axiom, namely xPy and $yPx \Rightarrow x = y$. We may consider, in a way to be described, that a partial ordering induces a form of 'equivalence' relation with a P-like ordering between the resulting 'equivalence' classes. Regard the partial ordering P as a means of distinguishing between elements of the set it orders, then those elements which cannot be distinguished from one particular element by P will be said to be *P-wise indistinguishable* from the specified element. However, we must notice that the *indistinguishability classes*—as we shall name the 'equivalence' classes induced by P—are not necessarily true equivalence classes, for no non-trivial transitivity within the classes is necessarily defined, and it is certainly impossible for the P-transitivity to hold non-trivially.

An example of well-defined indistinguishability classes which give rise to a partition, is the set of lines $y =$ constant in the two-dimensional cartesian plane, with the partial order $(x,y_1) \leqslant (x',y_2)$ IFOF $y_1 \leqslant y_2$. An example of a set of indistinguishability classes which do not give rise to a partition is the set of light cones associated with the points along a time-like path in Minkowski space; here the partial ordering between two points x,y is given by

$$(y_0 - x_0)^2 \geqslant (y_1 - x_1)^2 + (y_2 - x_2)^2 + (y_3 - x_3)^2$$

4.4. *Homotopy of Partial Ordering Commutators*

Interpreting a partial ordering R as a collection of ordering operators as in Section 4.2 is clearly possible, but the collection of all the operators will have a different structure. Let us give a simple examination of the case where the associated indistinguishability classes are well defined. Then we may show, as below, that the ordering operators between pairs of elements in a single equivalence class clearly are all deformable one into the other owing to the symmetric condition on the ordering relation in the class. This, then, means that there are homotopy classes of ordering operators corresponding to the R-wise (indistinguishability) equivalence classes. However, there are also ordering operators taking an element of one class into an element of another class, and since they relate distinct classes they (the operators) must be distinct from one another. If one considers these ordering operators as maps between classes, then they have no internal structure because they are maps between single elements; but if they are considered as being quantities which characterise all possible maps between pairs of elements from two equivalence classes, then they can be seen to have the structure of homotopy classes. For let, X, Y be two equivalence classes, $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ with $x_1 R^{XY}_{(11)} y_1$ and $x_2 R^{XY}_{(22)} y_2$;

also let $R^{(0)}_{X(ij)}$, $R^{(0)}_{Y(kl)}$ denote the ordering operators in R-wise indistinguishability classes—they are homotopy classes of operators as was remarked earlier. Then we may write:

$$x_1 R^{XY}_{(11)} y_1, \qquad y_1 R^{(0)}_{Y(12)} y_2 \Rightarrow x_1 R^{XY}_{(11)} R^{(0)}_{Y(12)} y_2 \Rightarrow R^{XY}_{(11)} \simeq R^{XY}_{(11)} R^{(0)}_{Y(12)}$$

where the symbol $\simeq$ denotes the homotopy relationship. But we also have $x_1 R^{XY}_{(12)} y_2$, and since we do not assume that $R^{XY}_{(ij)}$ maps X onto Y via any intermediate equivalence classes we must assume† that $R^{XY}_{(12)} = R^{XY}_{(11)} R^{(0)}_{Y(12)} \simeq R^{XY}_{(11)}$. Similarly we may deduce that $R^{XY}_{(12)} \simeq R^{XY}_{(22)}$ by making use of the operators $R^{(0)}_{X(12)}$, $R^{XY}_{(22)}$. Hence, $R^{XY}_{(11)} \simeq R^{XY}_{(22)}$, and we may consider R^{XY} as symbolising either the homotopy class of ordering operators $R^{XY}_{(ij)}$ between $x_i \in X$ and $y_j \in Y$, or as the ordering operator between the class Y of R-wise indistinguishable elements. Clearly these two notions may be freely interchanged. We may therefore consider the partial ordering operator, associated with the partial ordering R, of a set, as the collection of all possible products of ordering operators such as $R^{(0)}_X$ and R^{XY}. The inverse partial ordering operator R^{-1} will consist of all possible products of ordering operators $R^{(0)}_X$ and $(R^{XY})^{-1}$; there can be no inverse defined for $R^{(0)}_X$ because the equivalence class X contains elements which cannot be separated into a sequence that is ordered by the sense of (i.e. direction associated with) R.

The products RR^{-1} and $R^{-1}R$, consequently, represent all possible products of an R^{XY} operation followed by an $(R^{ZY})^{-1}$ operation, and vice versa, to within some $R^{(0)}_X$, $R^{(0)}_Y$, $R^{(0)}_Z$ operations. Thus RR^{-1} would consist of all possible operator products $R^{(0)}_X R^{XY} R^{(0)}_Y (R^{ZY})^{-1} R^{(0)}_Z$. The commutator relation states that the loop products RR^{-1} and $R^{-1}R$ may be deformed into each other.

4.5. *Homotopy of Total Order Commutators*

A total ordering R in a set separates the set into discrete elements, that is to say every point is R-wise distinguishable from another, or if it is the maximal element it is R^{-1}-wise distinguishable from every other. (Notice that one may say that a partial ordering gives a form of total ordering to its 'indistinguishability' classes.) One can again interpret R as the collection of all possible products of (total) ordering operators between elements of the totally ordered set; similarly for R^{-1}.

The commutator relation for a total ordering states that the loops of RR^{-1} and $R^{-1}R$ are generally distinguishable, and hence a definite direction may be assigned to their 'tracks' (i.e. paths).

Since the ordering operators carrying indistinguishability classes into each other (for a partial ordering) are total orders, it is clear that the lack of definition of a sense of direction to the loops of a partial ordering, must arise from the lack of 'assignability' of direction to the ordering operators within the indistinguishability classes.

† This assumption must be made, because no other structure has been introduced which could allow any contradiction when tests for R-wise indistinguishability are made.

4.6. *Hyperbolic Metrics*

Consider Minkowski space M with the following structure owing to Zeeman (1964). Let $M = R^4$, be four-dimensional Euclidean space, and let Q denote the characteristic quadratic form on M given by:

$$Q(x) = x_0^2 - x_1^2 - x_2^2 - x_3^2, \qquad \text{where } x = (x_0, x_1, x_2, x_3) \in M$$

Let $<$ be the partial ordering on M given by $x < y$ if the vector $y - x$ is time-like, i.e. $Q(y - x) > 0$, and oriented towards the future, i.e. $x_0 < y_0$. Let G be the group of automorphisms of M given by;

(1) The Lorentz group (all linear maps leaving Q invariant).
(2) Translations.
(3) Dilatations (multiplication by a scalar).

Every element of G either preserves or reverses the partial ordering $<$, consequently the subset $G_0 \subset G$ which preserves $<$ is a subgroup of index

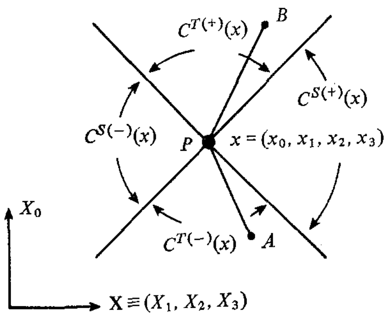


Figure 2.

two in G. These statements may be made as a result of the following theorem owing to Zeeman (1964):

TH(3): The group of automorphisms of M preserving the partial ordering $<$ is the group G_0. ▌

Given this structure, let us fit in the notion of ordering operator. Let us refer to Fig. 2. If $x \in M$, there are three cones through x which are invariant under G, namely:

DF(4): The *light cone* on x, $C^L(x) = \{y: Q(y - k) = 0\}$.

DF(5): The *time cone* on x, $C^T(x) = \{y: Q(y - x) > 0\}$.

DF(6): The *space cone* on x, $C^S(x) = \{y: Q(y - x) < 0\}$.

DF(7): A line through x is called a *light ray*, a *time axis*, or a *space line* according as to whether it passes through $C^L(x)$, $C^T(x)$, or $C^S(x)$ respectively.

If two points are separated by a time axis, e.g. the points A, B in Fig. 2, and $A < B$, then $Q(B - A) > 0$. Let us write this as follows in order to distinguish a time direction:

$$A < B \quad \text{is written} \quad Q(\overrightarrow{A};B) > 0$$
$$B > A \quad \text{is written} \quad Q(A;\overleftarrow{B}) > 0$$

If $Q(\overrightarrow{x};y) = s^2 > 0$, and we also make the interpretation $(\overrightarrow{x};y) = y - x$, that is the path goes from x towards y, and if similarly $Q(x;\overleftarrow{y}) = s^2 > 0$, with $(x;\overleftarrow{y}) = x - y = -(y - x)$ meaning that the path goes from y to x, then we can introduce two operators $\overrightarrow{Q}^{-1}: s \to (\overrightarrow{x};y) = y - x$, and $\overleftarrow{Q}^{-1}: s \to (x;\overleftarrow{y}) = x - y$. They plainly associate numbers s, related to the *interval* of the path, namely $s^2 = Q(x - y) = Q(y - x)$, with two points separated by that interval. To make these operators less ambiguous, let us consider $\overrightarrow{Q}^{-1}(s)$. $\overleftarrow{Q}^{-1}(s)$ as left operators in the sense of ordering operators and relations:

If $(\overrightarrow{x};y) = \overrightarrow{Q}^{-1}(s) = y - x$, then write $x\overrightarrow{Q}^{-1}(s)y$, or equivalently, $y = (x)(\overrightarrow{Q}^{-1}(s)) \equiv x\overrightarrow{Q}^{-1}(s)$.

If $(x;\overleftarrow{y}) = \overleftarrow{Q}^{-1}(s) = x - y$, then write $y\overleftarrow{Q}^{-1}(s)x$, or equivalently, $x = (y)(\overleftarrow{Q}^{-1}(s)) \equiv y\overleftarrow{Q}^{-1}(s)$.

This notation makes quite clear the way in which $y\overleftarrow{Q}^{-1}(s)x$ appears as the inverse of the relation $x\overrightarrow{Q}^{-1}(s)y$, which latter is the relation of x preceding y in a time sense with a separation of interval s^z: thus all possible points y lie on a hyperbolic shell within the forward light cone $C^{T(+)}(x)$.

Now suppose that $Q(a;b) = s_{ab}^2 > 0$, $Q(b;c) = s_{bc}^2 > 0$; together these imply $\overrightarrow{Q}(a;c) = s_{ac}^2 > 0$. In general, if a series of points of M are all connected so that each has its immediate predecessor in its respective backward time cone $C^{T(-)}$, then we can always write for example:

$$Q(\overrightarrow{a}; a\overrightarrow{Q}^{-1}(s_1{}^2 > 0)\,\overrightarrow{Q}^{-1}(s_2{}^2 > 0)\ldots\overrightarrow{Q}^{-1}(s_n{}^2 > 0)) > 0 \qquad (4.6.1)$$

That this is so can be verified easily by drawing a diagram illustrating the argument of Q; again, a diagram will show that any one arrow over a $\overrightarrow{Q}^{-1}(s^2 > 0)$ is reversed, then the relation above does not generally hold, neither does it generally hold if any of $s_i{}^2 < 0$. One can see that the ordering relation/operator $\overrightarrow{Q}^{-1}(s^2 > 0)$ is a counterpart of the partial ordering $<$ on M, but that it contains the extra structure donated by the parameter s^2 in the way already described. Also $\overrightarrow{Q}^{-1}(s^2 > 0)$ satisfies the transitivity axiom, as equation (4.6.1) clearly demonstrates.

Let us now show that $\overrightarrow{Q}^{-1}(s^2 > 0)$, as an operator associated with an ordering relation, a total ordering satisfying our causality condition and that the former fact requires points of M to possess Zeeman's fine ε-neighbourhoods. When the possibility of equality is dropped from the ordering relation, i.e. we consider $Q^{-1}(s^2 > 0)$, the resulting total order eliminates the ambiguity, arising from the light cone indistinguishability classes, by removing the light cones. That is, given a point x, the operator $\overrightarrow{Q}^{-1}(s^2 > 0)$ has range $C^{T(+)}(x)$, therefore the ordering relation is defined in $x \cup C^{T(+)}(x)$; the operator $\overleftarrow{Q}^{-1}(s^2 > 0)$ has range $C^{T(-)}(x)$, and therefore the associated ordering relation is defined in $x \cup C^{T(-)}(x)$. Consequently for every point x_i belonging to a strictly time ordered sequence† of points $\{x_i\}$ one needs to associate the cones-with-vertices $C^{T(-)}(x_i) \cup x_i \cup C^{T(+)}(x_i) = x_i \cup C^T(x_i)$ in order to discuss predecessor and successor relationships. However, in order to include a valid discussion of products of $Q^{-1}(s^2 > 0)$ operators with forward and backward arrows (which means one must be able to test whether the result of applying such a product to a point x gives a point y which is time-wise or space-wise separated from x) it is required that there is also associated the space cone $C^S(x)$ with the point x; i.e. for a full discussion of the use of the $\overrightarrow{Q}^{-1}(s^2 > 0)$, $\overleftarrow{Q}^{-1}(s^2 > 0)$ operators, to every point x in M we must associate $x \cup C^T(x) \cup C^S(x)$. It only remains to make the convention that if a product $\prod (\overrightarrow{Q}^{-1}, \overleftarrow{Q}^{-1})$ is applied to a point x, then it gives a point $y \notin x \cup C^T(x) \cup C^S(x)$ IFOF $y \in C^L(x)$; in which case $\prod (\overrightarrow{Q}^{-1}, \overleftarrow{Q}^{-1})$ may be considered as some form of 'null product'. Making these considerations local, we have to associate to a point x a *fine ε-neighbourhood* $N_\varepsilon^F(x) \equiv N_\varepsilon^E(x) \cap (C^T(x) \cup C^S(x))$, where $N_\varepsilon^E(x)$ is a Euclidean ball of radius ε on the centre x; $N_\varepsilon^F(x)$ is just the Euclidean ε-ball with its light cone removed and its centre replaced, and is Zeeman's fine ε-neighbourhood (Zeeman, 1964).

It is now possible to demonstrate that the ordering operator $\overrightarrow{Q}^{-1}(s^2 > 0)$ satisfies the total order commutation relation, Section 2.5, PR(3), in its simple form. Consider the term $\overrightarrow{Q}^{-1}\overleftarrow{Q}^{-1}$; it may be represented by the closed loop in Fig. 3. Starting from A the operator $\overleftarrow{Q}^{-1}(s^2 > 0)$ translates A into the later point B by means of a time-like path. Since the general ordering operator can relate any two points time-wise separate, we may consider B to vary over the whole of the forward light cone of A. Similarly the term $\overleftarrow{Q}^{-1}\overrightarrow{Q}^{-1}$ in the commutator bracket will generate a loop in the backward light cone of A. The non-vanishing of the commutator requires, therefore that loops in the forward and backward light cones of a point cannot be deformed one into the other. The proof is as follows:

The forward and backward loops could be deformed one into the other if it were possible for them both to be contracted to the point A. Technically

† That is to say $x_{i-1} < x_i$ for all adjacent pairs of points.

speaking this would mean that loops in the forward and backward cones would be homotopic to the same constant map on A; however, in point of fact this would introduce difficulties concerning relative orientation of the loops besides the question of the structure of the loops. However, neither of the loops can be contracted to the constant map on A, and this can be shown by considering Fig. 3. As $B \to A$ the hyper-volume enclosed

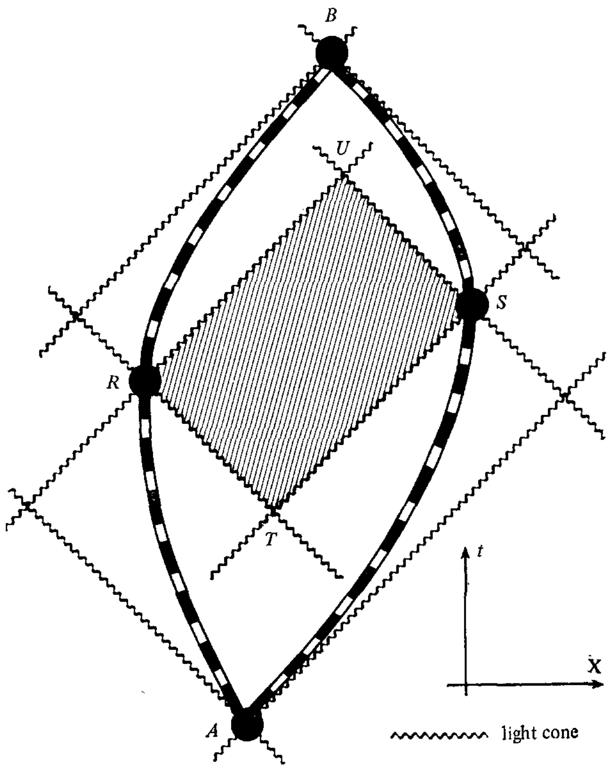


Figure 3.

by the loop will contract. In particular consider the two arbitrary points R, S, along the outgoing and incoming paths AB, BA, when the loop is very small. Then the fine neighbourhoods of R and S will define a region of inaccessibility for the loop $ARBS$, which is the shaded region. Moreover, the loop $ARBS$ is trapped in the region

$$D \equiv [G^{T(+)}(A) \cap G^{T(-)}(B)] - [C^{S(+)}(R) \cap G^{S(-)}(S)]$$

which becomes progressively smaller. Since by hypothesis the operator $\vec{Q}^{-1}(s^2 > 0)$ is total and cannot create a light path—i.e. in the fine neighbourhoods the 'light cones minus vertex' are absent—the loop cannot vanish. The non-vanishing of the loop can also be shown by the following contradiction: the intersections $E^{(\pm)} = C^{L(\pm)}(R) \cap C^{L(\pm)}(S)$, here labelled by the letters U, T, are vacuous in the fine neighbourhood topology used here; if $B \rightarrow A$ and B meets A, then A and T coincide and B and U must coincide, consequently the points A, B are members of empty sets—a condition clearly impossible except in a vacuous sense. As final remarks we may mention: (a) it does not matter what orientation is chosen for the loop *ARBS*; (b) this proof is still valid in curvilinear coordinates, for we have considered nothing more detailed than regions between light cones. We have therefore proved the following most important and satisfying theorem:

TH(4): The hyperbolic metric of Minkowski space generates an ordering relation that is causal. ∎

RMK(4): It is as well to notice that this result also acts as an existence theorem for metric representations of causal ordering relations/operators satisfying the commutator condition $[R, R^{-1}] \neq 0$.

4.7. *Topology Induced by a Causal Ordering*

The construction that has just been introduced, namely Zeeman's fine topology, does more than provide a useful example enabling a solution of the condition $[R, R^{-1}] \neq 0$ to be found. It indicates the fundamental nature of any topology induced upon a space by the causal operator which acts upon that particular space; this is that a form of 'disconnecting' must be introduced into the space: it is disconnecting in the sense that the cut prevents access of paths.

Considering Fig. 3, one can see that the light cones (or rather their non-existence) give rise to the non-contractibility of the loop *ARBSA*. The disconnecting cut must pass *through* the point, because otherwise there is a possible situation in which the forward and return paths may be deformed one into the other. For example in Fig. 3 if the sections of light cone *UST* and *URT* were moved so that their vertices were to the left of R and right of S respectively, then *ARB* could be deformed into *BSA*.

Since the ordering relation is transitive, the 'discontinuity' associated with a point cannot be of such a kind that it separates its predecessors into regions inaccessible from the point. Nor must the successors be separated into 'disconnected' regions if transitivity is to be preserved by the induced topology. Therefore the topology induced by a series of points ordered by the relation gives rise to a tube-like region of the space within which the path lies.

Moreover, the 'disconnection' must bear a symmetry (about the point it is associated with) that is determined by the ordering group $\check{\mathbf{R}}$.† There are two, coupled, reasons for this: (1) successors of the point are accessible

† This notation is taken from the more formal approach developed in Part III, but one can perfectly well think of $\mathbf{R}_0$.

within the 'tube' by means of 'transitions' which have the structure of $\check{\mathbf{R}}$; (2) the inaccessible region within a loop (see Fig. 3) is governed in shape by the deformations of the constituent paths of the loop and these are given by $\check{\mathbf{R}}$. We can therefore state:

PR(5): A causal ordering in a space induces a topology of which the neighbourhoods of points have the following properties:

(1) There is a disconnecting cut in the neighbourhood.
(2) The 'disconnection' meets the base point considered.
(3) The region within which dynamical transitions may occur is (open-ended) tube-like.
(4) The tube and the 'disconnection' defining the tube have the symmetry of the ordering operator group.]

RMK(5): Light cones clearly have these properties, for they define the 'disconnection' in fine neighbourhoods and have funnel shaped 'tubes'.

RMK(6): It is also clear that if the three-space of the space-time in which the causal ordering operator acts is locally metrisable, then the 'metric' associated with the ordering operator must be 'at least' quadratic and hyperbolic, otherwise there could be no 'light cones' giving rise to the open-ended tube.

5. *Conclusions*

There has first been presented an intuitive way of thinking of an ordering relation as an operator, more precisely, as a family of operators. Conditions have also been found that relations of various degrees of strength must satisfy. In particular the commutation relation for the total ordering operator has been seen to be of use. Physics has been considered as a study that must be subject to the Principle of Corporate Agreement, a principle defined here for the purpose of arriving at a metric-free notion of causality. This principle will turn out to be of significance in the study of quotient structures in Part III of this series. Lastly an existence theorem for the commutator condition on a (total) causal ordering has been proved, and it shows that spaces which are used for the coordinate-cartographical description of causally related events must have 'at least' a hyperbolic metric.

5.1. *Further Remarks*

The interpretation of an ordering relation as an operator given here is rather inadequate, firstly because it hardly touches any mathematical foundation, secondly because it does not relate to observation, the basic material of any study of physics. These two defects are treated on a much more serious level in Part III.

Also we have not made any mention of causality in classical non-relativistic dynamics, which uses ordinary Euclidean space. An apparent contradiction is therefore presented if one believes that causality can be fully described by dynamics in Euclidean space as well as in Minkowski space. Here we have implied that Minkowski space is acceptable but that Euclidean space is not. No full discussion on this matter can be properly entered into until after all the considerations of Part III have been presented that relate to fundamental groups.

Acknowledgements

I must offer considerable thanks to Professor C. W. Kilmister of King's College, London W.C.2, for encouragement and criticism. Thanks are due to Professor D. Bohm of Birkbeck College, London W.C.1, for some early discussion on the relationship between causality and observers. Also I must thank my former colleagues of the Post Office Research Station, Dollis Hill, London N.W.2, for their patience and consideration whilst I tried to clarify ideas through discussion.

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